

Zeta-function on the generalised cone

Guido Cognola ^{*} and Sergio Zerbini [†]

Dipartimento di Fisica, Università di Trento
and Istituto Nazionale di Fisica Nucleare,
Gruppo Collegato di Trento, Italia

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Abstract: *The analytic properties of the ζ -function for a Laplace operator on a generalised cone $\mathbb{R}^+ \times \mathcal{M}^N$ are studied in some detail using the Cheeger's approach and explicit expressions are given. In the compact case, the ζ -function of the Laplace operator turns out to be singular at the origin. As a result, strictly speaking, the ζ -function regularisation does not “regularise” and a further subtraction is required for the related one-loop effective potential.*

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Manifolds with conical singularities attracted the interest of some physicist since the beginning of the century with the works of Sommerfeld, but it was in the last decades that they become popular among all physicists working on space-times with horizons. The reason is due to the fact that in such kind of space-times there is a natural equilibrium temperature, the Hawking temperature, which, within the Euclidean approach, can be easily computed by imposing the absence of the conical singularity [1], i.e. by requiring the space-time to be a smooth manifold. A lot of work has been done in this direction, mainly concerning the pure cone, where heat kernel [2, 3, 4, 5, 6, 7] and ζ -function [8, 9, 10] and their applications to physics have been studied in some detail. The generalised cone has been investigated for the first time in a seminal paper by Cheeger [11], where the interested reader can find the general properties concerning the heat kernel and ζ -function related to the Laplace operator on functions and more recently by Bordag, Dowker and Kirsten [12, 13], where the generalisation to generic p -forms has also been carried out. Here we derive explicit analytic expressions for the ζ -function of a Laplace operator acting on functions (massless scalar fields) in a generalised cone with an arbitrary smooth base, following the Cheeger's approach [11].

To start with we remind that the $D = N + 1$ dimensional generalised cone $\mathcal{M}^D = \mathbb{R}^+ \times \mathcal{M}^N$ has local properties described by the metric

$$ds^2 = dr^2 + r^2 d\sigma_N^2, \quad (1)$$

where $d\sigma_N^2$ is the metric of the compact smooth manifold \mathcal{M}^N , with or without boundary (the base). Let us denote $x = (r, \tau) \in \mathcal{M}^D$, $\tau \in \mathcal{M}^N$. For example, for the pure cone with deficit angle $2\pi - \beta$, $0 < \tau < \beta$, β being a parameter which takes the conical singularity into account. If $\beta = 2\pi$ then $\mathcal{M}^1 \equiv S^1$ and in this case \mathcal{M}^2 is a smooth manifold. If $r \in R^+$, the generalised

^{*}e-mail: cognola@science.unitn.it

[†]e-mail: zerbini@science.unitn.it

cone is non-compact and the spectral properties of the Laplace operator on $\mathcal{M}^D = \mathbb{R}^+ \times \mathcal{M}^N$ are well known, the spectrum is continuous and a complete set of normalised eigenfunctions of the operator (negative Laplacian)

$$L_D = -\frac{\partial^2}{\partial r^2} - \frac{N}{r^2} \frac{\partial}{\partial r} - \frac{1}{r^2} \Delta_N, \quad (2)$$

is easily found to be

$$\psi_{\lambda\alpha}(r, \tau) = r^{\frac{1-N}{2}} J_{\nu_\alpha}(\lambda r) \phi_\alpha(\tau). \quad (3)$$

Here λ^2 ($\lambda \geq 0$) is the continuous eigenvalue corresponding to $\psi_{\lambda\alpha}$, while J_{ν_α} is the regular Bessel function. Moreover, Δ_N is the Laplace operator acting on functions in \mathcal{M}^N , then it has a discrete spectrum with eigenvalues λ_α^2 and eigenvectors ϕ_α . We have set $\nu_\alpha^2 = \lambda_\alpha^2 + \rho_N^2$, where $\rho_N = (N-1)/2$. With regard to the behaviour near the conical singularities, the compact case is similar to the non compact one, then we shall approximate it with the latter (for which we know the spectrum), but with the restriction $0 < r < R$.

For the diagonal kernel of a generic operator $F(L_D)$ one has [11]

$$F(r, \tau|L_D) = \sum_\alpha \int_0^\infty F(\lambda^2) |\psi_{\lambda\alpha}|^2 \lambda d\lambda. \quad (4)$$

Integrating on the transverse coordinates we obtain the reduced trace $F(r|L_D)$ on \mathcal{M}^N in the form

$$F(r|L_D) = \frac{1}{r^D} \sum_\alpha \int_0^\infty F\left(\frac{\lambda^2}{r^2}\right) J_{\nu_\alpha}^2(\lambda) \lambda d\lambda. \quad (5)$$

As it stands, such an expression is only formal, since the series and the integral could not be convergent.

Since we are mainly interested in the ζ -function, we choose $F(L_D) = L_D^{-s}$ and, using Eq. (5), we formally have

$$\zeta(s; r|L_D) = r^{2s-D} \sum_\alpha \int_0^\infty \lambda^{1-2s} J_{\nu_\alpha}^2(\lambda) d\lambda. \quad (6)$$

The integration over λ can be performed providing that $\frac{1}{2} < \text{Re } s < \text{Re } \nu_\alpha + 1$, while the series in α converges if $\text{Re } s > D/2$. These restrictions have a non vanishing intersection for any α if $\nu_\alpha > \rho_N$. If such a condition is not satisfied (this is the more common case), one has to treat separately low and high eigenvalues, considering $\nu_\alpha \leq \nu_{\tilde{\alpha}} \leq \rho_N$ and $\nu_\alpha > \nu_{\tilde{\alpha}}$ as in the paper of Cheeger [11]. Thus we write

$$\begin{aligned} \zeta_{<}(s; r|L_D) &= r^{2s-D} \frac{\Gamma(s - \frac{1}{2})}{\sqrt{4\pi}\Gamma(s)} \sum_{\alpha \leq \tilde{\alpha}} \frac{\Gamma(\nu_\alpha - s + 1)}{\Gamma(\nu_\alpha + s)}, \\ \zeta_{>}(s; r|L_D) &= r^{2s-D} \frac{\Gamma(s - \frac{1}{2})}{\sqrt{4\pi}\Gamma(s)} \sum_{\alpha > \tilde{\alpha}} \frac{\Gamma(\nu_\alpha - s + 1)}{\Gamma(\nu_\alpha + s)}, \end{aligned} \quad (7)$$

and, after the analytic continuation, we may define the ζ -function as the sum of $\zeta_{<}$ and $\zeta_{>}$.

We anticipate the final result, which reads

$$\begin{aligned} \zeta(s; r|L_D) &= \zeta_{<}(s; r|L_D) + \zeta_{>}(s; r|L_D) = r^{2s-D} \frac{\Gamma(s - \frac{1}{2})}{\sqrt{4\pi}\Gamma(s)} [G(s) + \mathcal{N}\Gamma(1-s)], \\ &= r^{2s-D} \frac{\Gamma(s - \frac{1}{2})}{\sqrt{4\pi}\Gamma(s)} \left[\sum_{j=0}^\infty c_j(s) \zeta(s + j - \frac{1}{2}|L_N) + F(s) + \mathcal{N}\Gamma(1-s) \right], \end{aligned} \quad (8)$$

where $L_N = -\Delta_N + \rho_N^2$, \mathcal{N} represents the number of zero modes of L_N , while the properties of $G(s)$, $F(s)$ and $c_j(s)$ will be studied in some detail in the following. As we shall see, $F(0) = 0$, $c_0(s) = 1$, while $c_j(0) = 0$ for any $j > 0$. Furthermore, the meromorphic structure of $\zeta(s|L_N)$ is given by Seeley theorem [14] and reads

$$\zeta(s|L_N) = \frac{1}{\Gamma(s)} \left[\sum_{n=0}^{\infty} \frac{K_n(L_N)}{s - \frac{N-n}{2}} + \text{analytic term} \right], \quad (9)$$

$K_n(L_N)$ being the well known Seeley-De Witt coefficients. As a consequence, from Eq. (8) we immediately get

$$\zeta(0; r|L_D) = r^{-D} \frac{K_D(L_N)}{\sqrt{4\pi}}. \quad (10)$$

In the non compact case the integration of the latter equation over r , with the measure $r^N dr$ requires two cutoffs for small and large r respectively. For the trace at $s = 0$ one gets

$$\zeta(0|L_D) = \int_{\varepsilon}^{\Lambda} \zeta_a(0; r|L_D) r^N dr = \frac{K_D(L_N)}{\sqrt{4\pi}} \ln \frac{\Lambda}{\varepsilon}, \quad (11)$$

namely it is logarithmically divergent. The same result can be obtained first integrating $\zeta(s; r|L_D)$ with respect to r and then taking the limit $s \rightarrow 0$. However, in the compact case we may perform the integration considering $\text{Re } s$ sufficiently large, in order to have the convergence at $r = 0$. Thus, near $s = 0$ we have

$$\begin{aligned} \zeta(s|L_D) &= \frac{R^{2s} \Gamma(s - \frac{1}{2})}{2\sqrt{4\pi} \Gamma(s + 1)} [G(s) + \mathcal{N} \Gamma(1 - s)] \\ &= \frac{K_D(L_N)}{2\sqrt{4\pi}} \frac{1}{s} + \frac{K_D(L_N)}{\sqrt{4\pi}} \ln R - \frac{\mathcal{N}}{2} + O(s). \end{aligned} \quad (12)$$

This result must be compared with Ref. [12], where the special case in which M^N is a sphere of radius a has been analyzed. This is our main result. In the compact generalised cone the ζ -function of the Laplace operator may have a pole at $s = 0$, in contrast with the Minakshisundaram theorem [15], which states that the ζ -function is regular in the smooth compact case. In the non compact case one gets a logarithmic divergence. Furthermore, if the base of the generalised cone \mathcal{M}^N is an even-dimensional, smooth manifold without boundary, then $K_D(L_N) = 0$ and the ζ -function has the usual meromorphic structure. This is also trivially true for the pure (2-dimensional) cone case, since the base is a flat manifold. If \mathcal{M}^N has boundary, then the pole at $s = 0$ is always present.

Alternatively, the singularity at $s = 0$ of the ζ -function could be traced back from the asymptotics of the heat-kernel trace. In fact one has

$$\text{Tr } e^{-tL_D} = \frac{1}{2\pi i} \int_{\text{Re } s > D/2} \Gamma(s) t^{-s} \zeta(s|L_D) ds. \quad (13)$$

Shifting the vertical contour to the left one gets the asymptotic expansion for short t and in particular the pole of the second order at $s = 0$ gives rise to a logarithmic term in t proportional to $K_D(L_N)$, in agreement with the Cheeger's result [11]. Conversely, if one has a logarithmic term in t in the heat-kernel expansion, the presence of a simple pole at $s = 0$ in the ζ -function directly follows (see for example Ref. [12]). In a different setting, this happens if one is dealing with a scalar massive field on the hyperbolic space-time $R \times H^3/\Gamma$, H^3/Γ being a non compact hyperbolic manifold with finite volume [16].

Now we outline the procedure leading to the analytic continuation of the ζ -function at $s = 0$. Eq. (8) implies that it is sufficient to deal with function $G(s)$. Thus the starting point is the series

$$G(s) = \sum_{\nu_\alpha \neq 0} \frac{\Gamma(\nu_\alpha - s + 1)}{\Gamma(\nu_\alpha + s)}, \quad (14)$$

which has been introduced in Ref. [11]. In Eq. (14) $\nu_\alpha^2 = \lambda_\alpha^2 + \rho_N^2$. We recall that $L_N = -\Delta_N + \rho_N^2$, Δ_N being the Laplace operator acting on functions in \mathcal{M}^N and $\rho_N = (N-1)/2$. With regard to the convergence of the latter series we observe that for $\nu \rightarrow \infty$ one has the asymptotic expansion

$$\frac{\Gamma(\nu - s + 1)}{\Gamma(\nu + s)} \sim \nu^{1-2s} \sum_{j=0}^{\infty} c_j(s) \nu^{-2j} \quad (15)$$

and from Weyl's theorem the degeneracy of the eigenvalues behaves asymptotically as ν_α^N . As a result, the series in Eq. (14) is convergent for $\text{Re } s > (N+1)/2$. The $c_j(s)$ coefficients in Eq. (15) are computable using the asymptotic expansion of $\Gamma(z)$ for large z , namely

$$\Gamma(z) \sim \sqrt{\frac{2\pi}{z}} e^{-z+z \ln z + B(z)}, \quad B(z) = \sum_{j=0}^{\infty} \frac{B_{2j} z^{1-2j}}{2j(2j-1)},$$

B_j being the Bernoulli numbers. It is easy to see that the function $\frac{\Gamma(\nu-s+1)}{\Gamma(\nu+s)}$ for any $s = -n/2$ ($n = -1, 0, 1, 2, \dots$) is effectively a polynomial of order ν^{n+1} , in fact

$$\frac{\Gamma(\nu + n/2 + 1)}{\Gamma(\nu - n/2)} = \begin{cases} \nu [\nu^2 - 1] [\nu^2 - 2^2] \cdots [\nu^2 - (\frac{n}{2})^2], & n = 0, 2, 4, \dots \\ \left[\nu^2 - \left(\frac{1}{2}\right)^2 \right] \left[\nu^2 - \left(\frac{3}{2}\right)^2 \right] \cdots \left[\nu^2 - \left(\frac{n}{2}\right)^2 \right], & n = -1, 1, 3, \dots \end{cases} \quad (16)$$

Then it follows that $c_j(-n/2)$ must vanish for all $j > (n+1)/2$ (they have a simple zero at $s = -n/2$, $n = -1, 0, 1, 2, \dots$). One can directly verify that also $c_j(1) = 0$ for all $j > 0$. The first coefficients can be computed and read

$$c_0(s) = 1, \quad c_1(s) = \frac{s(s-1/2)(s-1)}{3}; \quad c_2(s) = \frac{s(s^2-1/4)(s^2-1)(s-6/5)}{18}. \quad (17)$$

It has to be noted that $G(s)$ has a simple poles at $s = \frac{N+1}{2}$ and is certainly analytic for $\text{Re } s > \frac{N+1}{2}$. In order to make the analytic continuation for any s we define

$$f_{\tilde{n}}(\nu, s) = \frac{\Gamma(\nu - s + 1)}{\Gamma(\nu + s)} - \sum_{j=0}^{\left[\frac{\tilde{n}}{2}\right]+1} c_j(s) \nu^{1-2s-2j} \sim c_{\left[\frac{\tilde{n}}{2}\right]+2}(s) \nu^{-(2s+2\left[\frac{\tilde{n}}{2}\right]+3)},$$

where $\left[\frac{\tilde{n}}{2}\right]$ represents the integer part of $\frac{\tilde{n}}{2}$. For $\text{Re } s > \frac{N+1}{2}$ we have

$$G(s) = \sum_{j=0}^{\left[\frac{\tilde{n}}{2}\right]+1} c_j(s) \zeta(s+j-\frac{1}{2}|L_N) + F(s), \quad F(s) = \sum_{\alpha} f_{\tilde{n}}(\nu_\alpha, s). \quad (18)$$

Now, the right hand side of the latter equation has meaning for $\text{Re } s > \frac{N-3}{2} - \left[\frac{\tilde{n}}{2}\right]$ and so we have obtained the analytic continuation we were looking for. It is interesting to observe that the functions $f_{\tilde{n}}(\nu, s)$, for a sufficiently large \tilde{n} , are identically vanishing at $s = 1, 1/2, 0, -1/2, -1, \dots$ and as a consequence also $F(s)$ is vanishing in all that points. This fact permits us to compute the

behaviour of $G(s)$ in a neighbourhood of the half-integer points $s = -n/2$ ($n = -2, -1, 0, 1, 2, \dots$). In particular, near $s = 0$ one obtains

$$G(s) = \frac{K_{N+1}(L_N)}{\Gamma(-1/2)s} + \tilde{\zeta}(-\tfrac{1}{2}|L_N) + \left| \frac{K_{N-1}(L_N)}{\Gamma(1/2)} \frac{c_1(s)}{s} + \frac{K_{N-3}(L_N)}{\Gamma(3/2)} \frac{c_2(s)}{s} + \dots \right|_{s=0}, \quad (19)$$

$K_n(L_N)$ being the spectral coefficients and $\tilde{\zeta}(s_0|L_N)$ the finite part of the ζ -function at s_0 .

If \mathcal{M}^N is a smooth manifold without boundary, then all spectral coefficients with odd n are vanishing. Thus, for even N (odd D), it follows that $G(0) = \zeta(-1/2|L_N)$, while for odd N (even D), the first term in the latter equation gives rise to the “anomalous” divergent contribution in Eq. (12).

As a simple example let us consider the pure cone with deficit angle $2\pi - \beta$. The eigenvalues of L_1 are $\lambda_\alpha = \nu_\alpha^2 = (2\pi\alpha/\beta)^2$, $\alpha \in \mathbb{Z}$ and moreover ρ_1 is vanishing as well as all K_n , but $K_0 = \frac{\beta}{\sqrt{4\pi}}$. Disregarding the null eigenvalue we get

$$\zeta(s|L_1) = \left(\frac{2\pi}{\beta}\right)^{-2s} \sum_{\alpha \in \mathbb{Z}, \alpha \neq 0} \alpha^{-2s} = 2 \left(\frac{\beta}{2\pi}\right)^{2s} \zeta_R(2s), \quad (20)$$

from which the well known result

$$G(0) = \frac{1}{6} \left(\frac{\beta}{2\pi} - \frac{2\pi}{\beta} \right). \quad (21)$$

directly follows. Here ζ_R is the usual Riemann’s zeta-function. Note that in Ref. [10] the function $G(s)$ is defined in a slightly different way.

We conclude with some remarks. In this letter the analytic properties of the ζ -function related to the Laplace operator on a generalised cone $\mathbb{R}^+ \times \mathcal{M}^N$ have been investigated using the Cheeger’s approach and explicit expressions for it have been obtained. We have shown that in the compact case, the ζ -function of the Laplace operator for a minimally coupled massless scalar field turns out to be singular at the origin. As a consequence, since the one-loop effective action in the ζ -function regularisation approach is formally given by $-\zeta'(0|L_D)/2$, looking at Eq. (12) one sees that a further subtraction is required in order to remove the singularity at $s = 0$. This singularity is proportional to the spectral coefficient $K_D(-\Delta_N + \rho_N^2)$, thus in principle, the nature of the counterterm is known. It has to be noted that if the base of the generalised cone is a manifold with boundary, also in the odd- D dimensional case the singularity at $s = 0$ of the ζ -function is present.

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